Finite element method versus finite difference method: Numerical accuracy study for two different applications

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Despite its wide-spread use in geology and geophysics, it is unclear whether there is a difference in accuracy between finite element (FEM) and finite difference methods (FDM). For this reason, we here compare the accuracy of the two methods for two different geophysical problems. In both cases an analytical solution is available that provides the exact solution to the respective problem and allows quantifying the numerical errors of the different methods.

The first study considers two-dimensional scattering of elastic waves in a medium containing a circular heterogeneity. Different combinations of the FDM and the FEM are used to approximate both time and space derivatives of the elastodynamic wave equation. The different numerical algorithms are compared for simulations of an incident plane P-wave that is scattered by a mechanically weak circular inclusion whereby the diameter of the inclusion is of the same order than the P-wave's wavelength. Staircase-like spatial discretization of the inclusion's circular shape with the FDM using a rectangular grid provides accurate velocity and displacement fields close to the inclusion boundary only for very high spatial resolutions. Implicit time integration based on either the FDM or the FEM does not provide computational advantages compared to explicit schemes. The best numerical algorithm in terms of accuracy and computation time for the investigated scattering problem consists of a FEM in space using an unstructured mesh combined with an explicit FDM in time.

The second study considers the two-dimensional pressure- and velocity field around a viscous circular inclusion embedded in a mechanically stiffer viscous matrix under pure shear boundary conditions. A number of different FDM and FEM are used to solve the Stokes equations. For the FDM, viscosity needs to be defined at different discrete points in a numerical control volume. The necessary viscosity interpolation is done in different ways, which yields differences in accuracy of up to one order of magnitude. In addition to the standard FDM, markers (i.e. marker-in-cell-technique) are used to carry the material parameters. Harmonic (in some cases geometric) averaging of the viscosity from markers to nodal points yields the most accurate results. Unstructured FEM with elements fitting exactly the material boundary are one to two orders of magnitude more accurate then Eulerian FDM or FEM. If viscosities are directly sampled at integration points of the finite elements, however, the FEM is less accurate than the FDM.

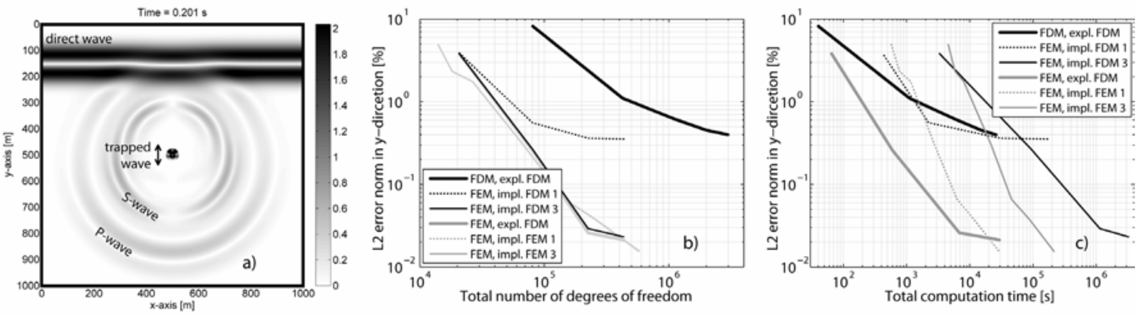
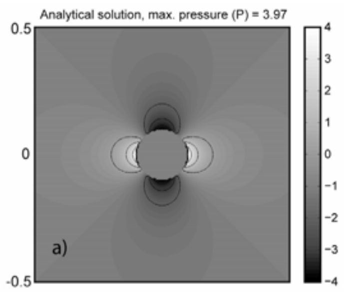


Figure 1: a) Snapshot of the simulated 2D wave field. Plotted is the normalized absolute value of the displacement field ($\sqrt{u_x^2 + u_y^2}$). A plane P-wave travels from bottom to top of the model and is scattered at the inclusion. b) and c) L2 error norm for particle displacement in y-direction versus total number of unknowns in the domain (b) and versus total computation time for the whole simulation (c). Different lines in b) and c) correspond to different numerical methods and/or different implicit time increments. Abbreviations in the legends before the comma (FDM or FEM) stands for the spatial discretization method, second abbreviation stands for the time discretization whereas expl. and impl. refers to explicit implicit time integration, respectively. Implicit time increments for both temporal FDM and temporal FEM are: 1: $\Delta t = 2.37 \cdot 10^{-4}$s, 3: $\Delta t = 2.96 \cdot 10^{-5}$s



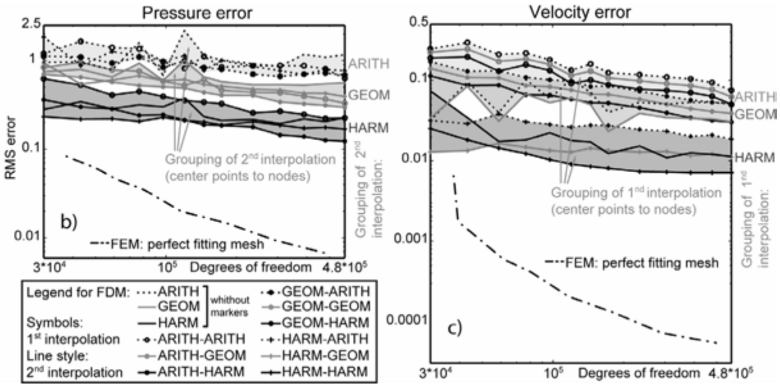


Figure 2: a) Normalized 2D pressure field around clast with pure-shear boundary conditions and viscosity contrast of μ_{clast}/μ_{matrix}=1000. b) and c) RMS error of pressure (b) and velocity (c) versus resolution for different viscosity interpolation methods. Displayed are the staggered grid FDM and the FEM with elements perfectly fitting the material boundary. Most FDM employ markers with viscosity first interpolated from markers to center points (marker2center), then interpolated from center to nodal points (center2node). In cases where no markers are employed (viscosity directly sampled at center points) only the second interpolation step is applied. Both interpolation steps can be performed with harmonic (HARM) or arithmetic (ARITH) averaging.

REFERENCES

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